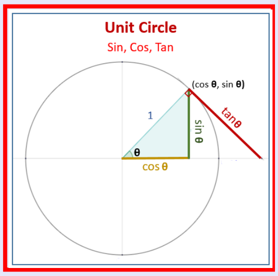
Math 241

Winter 2024

Lecture 12



Feb 19-8:47 AM

Class QZ

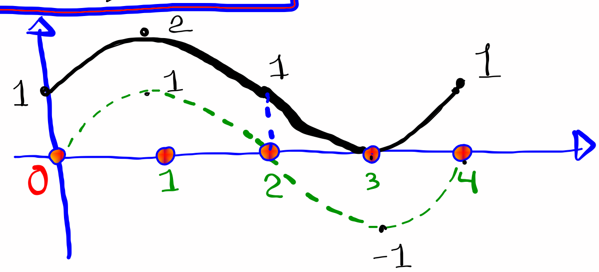
Graph $\boxed{y = \sin(.5\pi x) + 1}$

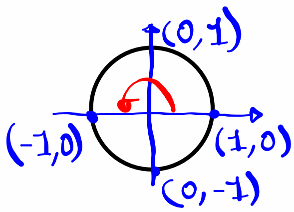
$0 \le .5\pi x \le 2\pi$

$0 \le \frac{1}{2}\pi x \le 2\pi$

$0 \le \pi x \le 4\pi$

$\boxed{0 \le x \le 4}$





Test Point $x = 2$

$y = \sin(.5\pi \cdot 2) + 1$

$= \sin(1\pi) + 1$

$= \sin\pi + 1 = 0 + 1 = 1$

Jan 22-11:56 AM

New identities:

Product – to – Sum

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

Sum – to product

$\sin A + \sin B$

$\sin A - \sin B$

$\cos A + \cos B$

$\cos A - \cos B$

Jan 22-11:52 AM

Write $4\cos 75^\circ \sin 25^\circ$ as sum or difference of two functions

$$4\cos 75^\circ \sin 25^\circ = 4\cdot\frac{1}{2}\left[\sin(75^\circ+25^\circ) - \sin(75^\circ-25^\circ)\right]$$

$$\cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$= 2\left[\sin 100^\circ - \sin 50^\circ\right]$$

write $6\cos 3x \cos x$ as sum or difference of two functions.

$$6\cos 3x \cos x = 6\cdot\frac{1}{2}\left[\cos(3x+x) + \cos(3x-x)\right]$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

$$= \boxed{3\left[\cos 4x + \cos 2x\right]}$$

write $2\sin\frac{\pi}{6}\cos\frac{\pi}{3}$ as sum or difference of two functions

$30^\circ + 60^\circ$ $\quad$ $30^\circ - 60^\circ = -30^\circ$

$$2\sin\frac{\pi}{6}\cos\frac{\pi}{3} = 2\cdot\frac{1}{2}\left[\sin\left(\frac{\pi}{6}+\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\right]$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$= 1\left[\sin 90^\circ + \sin(-30^\circ)\right]$$

$$= \sin 90^\circ - \sin 30^\circ = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

$\sin(-\alpha) = -\sin\alpha$

$\sin(-30^\circ) = -\sin 30^\circ$

Jan 23-8:08 AM

New identities:

Product - to - Sum

$\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

Sum - to product

$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$

$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$

$\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

Jan 22-11:52 AM

write $\sin 2x - \sin 4x$ as a product of two functions

$\sin 2x - \sin 4x = 2 \cos \frac{2x+4x}{2} \sin \frac{2x-4x}{2}$

$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$

$\sin(-\alpha) = -\sin\alpha$

$= 2 \cos 3x \cdot \sin(-x)$

$= \boxed{-2 \cos 3x \sin x}$

write $\cos 25^\circ + \cos 50^\circ$ as a product of two functions.

$\cos 25^\circ + \cos 50^\circ = 2 \cos \frac{75^\circ}{2} \cos \frac{-25^\circ}{2}$

$\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$

Recall $\cos(-\alpha) = \cos\alpha$

$= 2 \cos \frac{75^\circ}{2} \cos \frac{25^\circ}{2}$

Jan 23-8:26 AM

Half - Angle Identities:

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

choose + or − based on what quadrant is $\frac{A}{2}$ in.

$$= \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

Find Exact Value

$$\sin 15^\circ = \sin\frac{30^\circ}{2} \leftarrow A$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\sin 15^\circ = +\sqrt{\frac{1-\cos 30^\circ}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2-\sqrt{3}}}{2}}$$

why is this answer different from

$$\sin 15^\circ = \sin(45^\circ - 30^\circ)$$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$?

Jan 22-11:55 AM

Find exact value for tan 22.5°

$$22.5^\circ = \frac{45^\circ}{2} \leftarrow A$$

$$\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

$$\tan 22.5^\circ = \tan\frac{45^\circ}{2} = \frac{1-\cos 45^\circ}{\sin 45^\circ}$$

$$= \frac{1-\frac{\sqrt{2}}{2}}{\sqrt{2}/2} = \frac{2-\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{2\sqrt{2}-\sqrt{4}}{\sqrt{4}} = \frac{2\sqrt{2}-2}{2}$$

$$\frac{\cancel{2}(\sqrt{2}-1)}{\cancel{2}} = \boxed{\sqrt{2}-1}$$

Jan 23-8:46 AM

Find exact value of

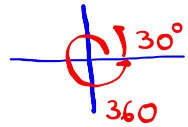
$\cos 195°$.

Hint: $195° = \frac{390°}{2} \leftarrow A$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\cos 195° = -\sqrt{\frac{1+\cos 390°}{2}}$$

Q III



$$= -\sqrt{\frac{1+\cos 30°}{2}}$$

$$= -\sqrt{\frac{1+\sqrt{3}/2}{2}}$$

$$= -\sqrt{\frac{2+\sqrt{3}}{4}}$$

$$= \boxed{-\frac{\sqrt{2+\sqrt{3}}}{2}}$$

Jan 23-8:52 AM

$195° = 150° + 45°$

$$\cos 195° = \cos(150° + 45°)$$

$$= \cos 150° \cos 45° - \sin 150° \sin 45°$$

$$= -\cos 30° \cos 45° - \sin 30° \sin 45°$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \boxed{-\frac{\sqrt{6}+\sqrt{2}}{4}}$$

why did we get $-\frac{\sqrt{2+\sqrt{3}}}{2}$ in using half-angle formula?

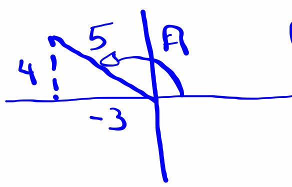
Jan 23-8:57 AM

$\sin A = \frac{4}{5} \quad 90^\circ < A < 180^\circ$ Find $\sin\frac{A}{2} = +$

$90^\circ < A < 180^\circ \rightarrow$ QII

$\frac{90^\circ}{2} < \frac{A}{2} < \frac{180^\circ}{2}$

$45^\circ < \frac{A}{2} < 90^\circ \rightarrow$ QI



$\cos A = \frac{-3}{5}$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$= +\sqrt{\frac{1-\frac{-3}{5}}{2}}$$

$$= \sqrt{\frac{1+\frac{3}{5}}{2}} = \sqrt{\frac{5+3}{10}}$$

LCD=5

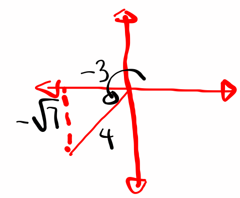
$$= \sqrt{\frac{8}{10}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$= \boxed{\frac{2\sqrt{5}}{5}}$$

Jan 23-9:03 AM

Given $\tan x = \frac{\sqrt{7}}{3} \quad 180^\circ < x < 270^\circ$

Find exact value of $\tan\boxed{\frac{x}{2}}$.



$180^\circ < x < 270^\circ$ $\quad \tan\frac{x}{2} = -$

$90^\circ < \frac{x}{2} < 135^\circ$

QII

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

$$\tan\frac{x}{2} = \frac{1-\frac{-3}{4}}{\frac{-\sqrt{7}}{4}} = \frac{4+3}{-\sqrt{7}} = \frac{7}{-\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}}$$

LCD=4

$$= \boxed{-\sqrt{7}}$$

Jan 23-9:11 AM

Verify $\sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$

$$\sin^2\frac{x}{2} = \left[\sin\frac{x}{2}\right]^2$$

$$= \left[\pm\sqrt{\frac{1-\cos x}{2}}\right]^2 = \frac{1-\cos x}{2} \cdot \frac{\tan x}{\tan x}$$

$$= \frac{\tan x - \overbrace{\cos x \cdot \tan x}^{\sin x}}{2\tan x}$$

$$\cos x \tan x = \cancel{\cos x} \cdot \frac{\sin x}{\cancel{\cos x}}$$

$$= \sin x$$

$$= \frac{\tan x - \sin x}{2\tan x} \checkmark$$

Jan 23-9:21 AM

Verify $\sin^2\frac{x}{2} = \frac{\tan x - \sin x}{2\tan x}$

$$\frac{\tan x - \sin x}{2\tan x} = \frac{\frac{\sin x}{\cos x} - \sin x}{\frac{2\sin x}{\cos x}} = \frac{\sin x - \sin x\cos x}{2\sin x}$$

LCD = $\cos x$

$$= \frac{\cancel{\sin x}(1-\cos x)}{2\cancel{\sin x}} = \frac{1-\cos x}{2} = \left[\pm\sqrt{\frac{1-\cos x}{2}}\right]^2$$

$$= \left[\sin\frac{x}{2}\right]^2 = \sin^2\frac{x}{2}$$

Jan 23-9:21 AM

Verify

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$$

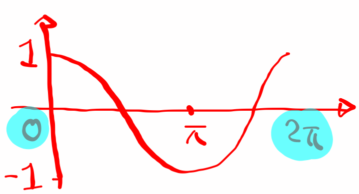
$$\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-\frac{1-\cos x}{1+\cos x}}{1+\frac{1-\cos x}{1+\cos x}} = \frac{\cancel{1}+\cos x-\cancel{1}+\cos x}{1+\cancel{\cos x}+1-\cancel{\cos x}}$$

$$\text{LCD} = 1+\cos x \qquad = \frac{\cancel{2}\cos x}{\cancel{2}} = \boxed{\cos x}$$

Jan 23-9:35 AM

Class QZ 9

Graph $y = -\cos(\pi x - \pi) - 1$ over one period.

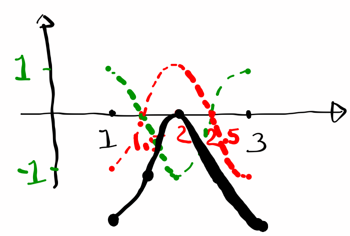


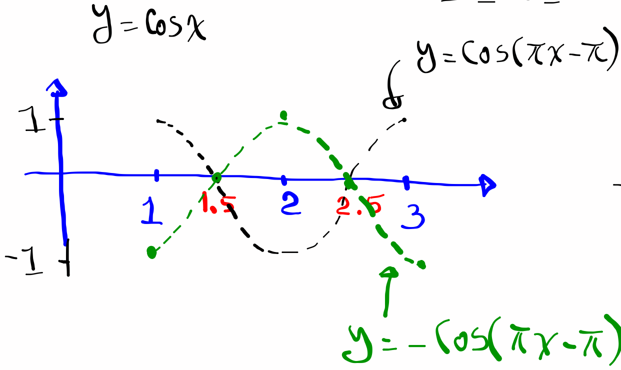
$y = \cos x$

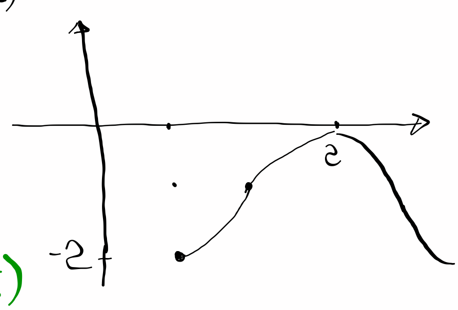
$$0 \le \pi x - \pi \le 2\pi$$

$$\pi \le \pi x \le 3\pi$$

$$1 \le x \le 3$$







Jan 23-9:43 AM

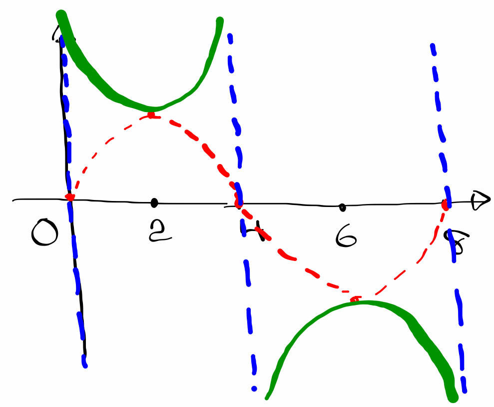
Graph $y = \csc\left(\frac{1}{4}\pi x\right)$

Graph $y = \sin\left(\frac{1}{4}\pi x\right)$

$0 \le \frac{1}{4}\pi x \le 2\pi$

$0 \le \pi x \le 8\pi$

$0 \le x \le 8$



Jan 23-10:17 AM

Graph $y = -\sec\left(\frac{1}{2}\pi x + 2\pi\right) + 2$

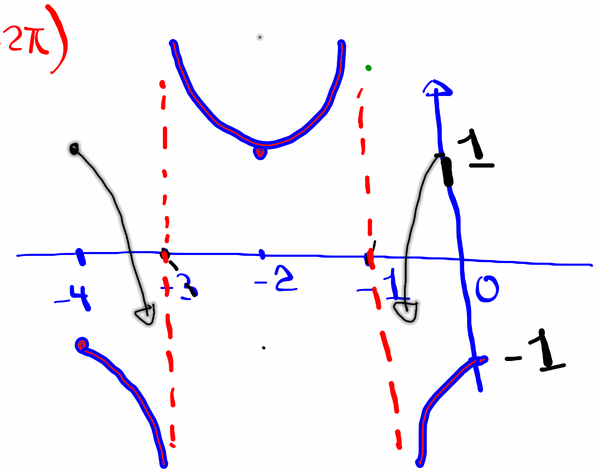
Graph $y = \cos\left(\frac{1}{2}\pi x + 2\pi\right)$

$0 \le \frac{1}{2}\pi x + 2\pi \le 2\pi$

$0 \le \pi x + 4\pi \le 4\pi$

$-4\pi \le \pi x \le 0$

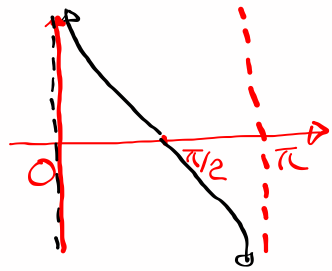
$-4 \le x \le 0$



Jan 23-10:27 AM

Graph $y = \cot\left(\frac{1}{2}\pi x - \frac{\pi}{2}\right)$

$y = \cot x$

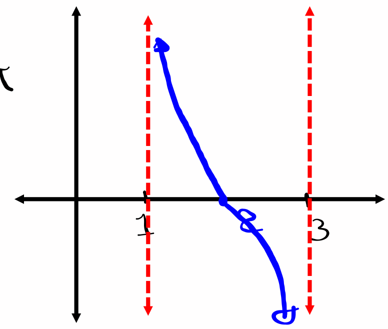


$0 < \frac{1}{2}\pi x - \frac{\pi}{2} < \pi$

$0 < \pi x - \pi < 2\pi$

$\pi < \pi x < 3\pi$

$1 < x < 3$



Find x: $\sin(6x-5)^\circ = \cos(4x+35)^\circ$ 37° 59°

Cofunctions & Compl. angles

$6x - 5 + 4x + 35 = 90 \longrightarrow \boxed{x=6}$

Jan 23-10:36 AM

Find x: $\tan\left(\frac{1}{2}x + 10\right)^\circ = \cot\left(\frac{1}{4}x - 10\right)^\circ$ 70° 20°

Equal Cofunctions → Compl. angles

$\frac{1}{2}x \cancel{+10} + \frac{1}{4}x \cancel{-10} = 90$

$\frac{3}{4}x = 90$

$3x = 4 \cdot 90$

$x = \frac{4 \cdot 90}{3}$ $\boxed{x = 120}$

$\frac{1}{2}x + 10 =$

$\frac{1}{2}(120) + 10 =$

$60 + 10 =$

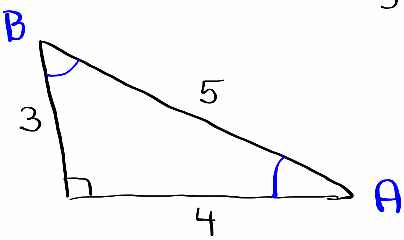
70

$\frac{1}{4}x - 10 =$

$\frac{1}{4}(120) - 10 = 30 - 10 = 20$

Jan 23-10:48 AM

Back to right - Triangle



$\sin A = \frac{3}{5} = .6$

$A = \underbrace{\sin^{-1}(.6)}_{\text{Sin inverse}}$

$\boxed{A \approx 37^\circ}$

$\sin B = \frac{4}{5} = .8$

$B = \sin^{-1}(.8)$

$\boxed{B \approx 53^\circ}$

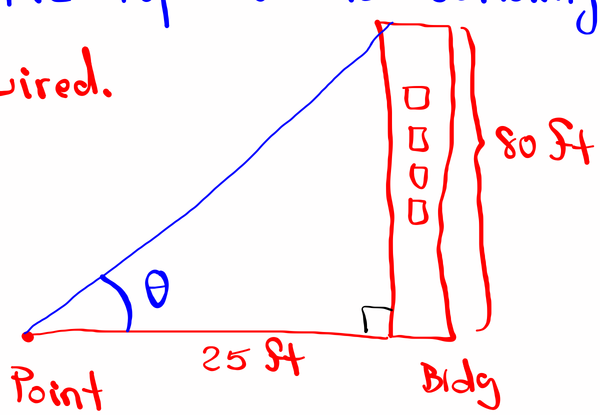
$\cos A = \frac{4}{5} = .8$

$A = \cos^{-1}(.8)$

$A \approx 37^\circ$

Jan 23-10:55 AM

From a point on the ground that is 25 ft from a tall building that 80 ft, find the measure of the elevation angle to the top of the building. Drawing required.



$\tan\theta = \frac{80}{25}$

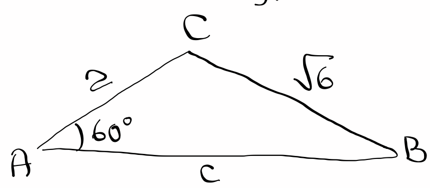
$\tan\theta = 3.2$

$\theta = \tan^{-1}(3.2)$

$\approx 73^\circ$

Jan 23-11:00 AM

Solve the triangle below



SSS
SAS → Law of Cosines

$$\boxed{\frac{\sqrt{6}}{\sin 60^\circ} = \frac{2}{\sin B}} = \frac{c}{\sin C}$$

$$\sqrt{6}\,\sin B = 2 \sin 60^\circ$$

$$\sqrt{6}\,\sin B = \cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}}$$

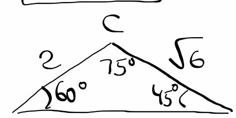
$$\sin B = \frac{\sqrt{3}}{\sqrt{6}} = \frac{\sqrt{3}}{\sqrt{3}\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

By Law of Sines

$$\sin B = \frac{\sqrt{2}}{2}$$

$$B = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\boxed{B = 45^\circ}$$



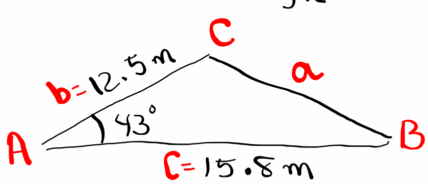
$$\frac{\sqrt{6}}{\sin 60^\circ} = \frac{c}{\sin 75^\circ}$$

⋮

$$c \approx 2.7$$

Jan 23-11:07 AM

Solve the triangle below



SAS

$$\boxed{\frac{10.8}{\sin 43^\circ} = \frac{12.5}{\sin B}} = \frac{15.8}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$a^2 = 12.5^2 + 15.8^2 - 2 \cdot 12.5 \cdot 15.8 \cdot \cos 43^\circ$$

$$\boxed{a \approx 10.8 \text{ m}}$$

$$\sin B = \frac{12.5 \cdot \sin 43^\circ}{10.8}$$

$$\sin B = .789$$

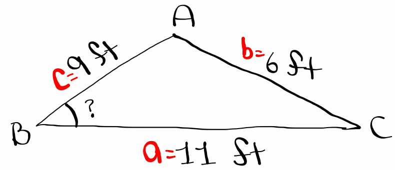
$$B = \sin^{-1}(.789)$$

$$\boxed{B \approx 52^\circ}$$

Jan 23-11:15 AM

Find angle B.

SSS $\rightarrow$ Law of Cosines



$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$2ac\cos B = a^2 + c^2 - b^2$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{11^2 + 9^2 - 6^2}{2(11)(9)}$$

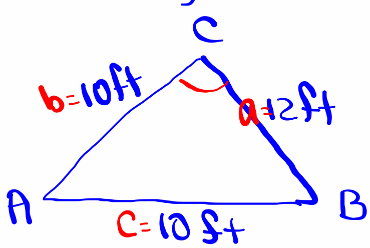
$$\cos B = .838$$

$$B = \cos^{-1}(.838)$$

$$\boxed{B \approx 33^\circ}$$

Jan 23-11:23 AM

Find angles B & C using triangle below



$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\vdots$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

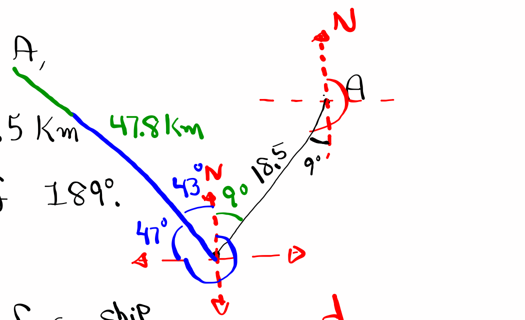
$$\cos C = \frac{12^2 + \cancel{10^2} - \cancel{10^2}}{2(12)(10)} = \frac{\cancel{12} \cdot \cancel{12}^{6}}{\cancel{2} \cdot \cancel{12} \cdot 10} = .6$$

$$\cos C = .6 \rightarrow C = \cos^{-1} .6 \rightarrow C \approx 53^\circ$$

$$B \approx 53^\circ$$

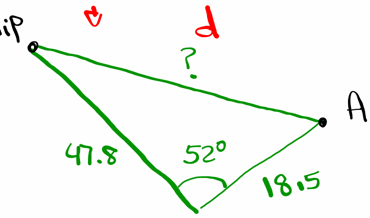
Jan 23-11:30 AM

Starting at point A, a ship sails 18.5 Km on a bearing of 189°.



It turns, sails for 47.8Km on a bearing of 317°

How far is the ship from point A?



SAS → Law of Cosines

$d^2 = 18.5^2 + 47.8^2 - 2 \cdot 18.5 \cdot 47.8 \cdot \cos 52°$

$\boxed{d \approx 39.2}$ Km

Jan 23-11:37 AM

Find the area of a triangle with sides 6cm, 8 cm, and 20cm.

In any triangle, sum of any two sides must be greater than the third side.

$8 + 20 > 6$

$6 + 20 > 8$

$6 + 8 > 20$ ← False   No such triangle

Heron's Formula $A = \sqrt{17(17-8)(17-6)(17-20)}$

$S = \frac{6+8+20}{2} = 17$ $= \sqrt{17 \cdot 9 \cdot 11 \cdot (-3)} = \sqrt{-\#}$

Not a real #

No such triangle

Jan 23-11:47 AM

Class QZ 10

Find the measure of both angles if

$$\tan \frac{1}{2}x = \cot \frac{1}{3}x$$

$\frac{1}{2}x + \frac{1}{3}x = 90$

LCD = 6

$3x + 2x = 6 \cdot 90$

$5x = 6 \cdot 90$ $\boxed{x = 108}$

$\frac{1}{2}(108) = \boxed{54^\circ}$

$\frac{1}{3}(108) = \boxed{36^\circ}$

Jan 23-11:57 AM